CETI Engineering Maths Notes

Linear Algebra

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Contents

1 General Resources

There are some excellent free MIT lectures on Linear Algebra here:
http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010

2 Matrix Decomposition

2.1 LU and Cholesky

Cholesky decomposition is for symmetric matrices, and $\mathbf{L} = \mathbf{U}^\top$. An algorithm for $\mathbf{LU}$ decomposition without pivoting is below (*i.e* this will not work for Question 3 of the examples sheet):

```
for k = 0 to n − 1 do
    for j = 0 to k + 1 do
        a_jk ← a_jk / a_kk;
        for j = 0 to k + 1 do
            a_ji ← a_ji − a_jk a_ki;
        end
    end
end
```

3 Solution of Linear Systems

3.1 Direct Methods

These methods involve performing Gaussian elimination to find the solution.

3.2 Indirect Methods

These methods involve providing an estimate of the solution and iterating to the correct one. Gauss Seidel will be used here, and the algorithm is as follows:

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=0}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n-1} a_{ij} x_j^{(k-1)} \right) \qquad (i = 0, 1, \cdots, n-1)$$

Keep performing the above calculation for the required number of iterations or until the residual reduces to a prescribed value.

3.3 Matrix Conditioning

A lot of methods are available for improving the conditioning of a matrix. This is important for indirect methods to both converge and solve a matrix quickly. The condition number of a matrix is calculated as follows (using norms):

$$\kappa = ||A||\ ||A^{-1}|| \tag{1}$$

It tells us the stability of a matrix when we perform operations such as $\mathbf{LU}$ decomposition. If it is large then finding the inverse may be difficult numerically. There is an example of this in your notes.

4 Supervision Activities

Let's use Ruby to decompose some matrices. There are various forms of the $\mathbf{LU}$ decomposition. Either $\mathbf{L}$ or $\mathbf{U}$ may have unity diagonals, or $\mathbf{L} = \mathbf{U}^\top$ and both have the same diagonal (known as Cholesky's decomposition and applied to square, symmetric matrices).

Q1 How could you work out the determinant of a matrix from its $\mathbf{LU}$ matrices? Code it.

Q2 What is the order of $\mathbf{LU}$ decomposition (*i.e.* how the number of operations scales with an increasing matrix size)? Try it on the following matrices:

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 3 & -6 & 0 \end{bmatrix}$$

Use Excel to plot the number of operations against the size of the matrix, and fit a power law curve to it.

Q3 *(Taken from 2006 P2 Q3)* Find the $\mathbf{LU}$ decomposition of the following matrix by hand and compare to the output of the Ruby program. What's the deteminant?:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Can you see how many solutions to this matrix there are for $\mathbf{A}.\mathbf{x} = [1, 1, c]^\top$, for specific values of c, just by examining it?

5 Examples Paper 3

Question 1

$$\begin{aligned} u + v + w &= 2 \\ u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 2 \\ 1 & 3 & 3 & 0 \\ 1 & 3 & 5 & 2 \end{array}\right]$$

Pivot around a_{11}:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & \mathbf{2} & 2 & -2 \\ 0 & 2 & 4 & 0 \end{array}\right]$$

Pivot around a_{22}:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

By back-substitution $w = 1$, $v = -2$ and $u = 3$.

Question 2

Part (a) $\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$

$$\begin{aligned} U_{11} &= 2 \\ U_{12} &= 1 \\ L_{21}U_{11} &= 8 \\ L_{21}U_{12} + U_{22} &= 7 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

Part (b) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

$$
\begin{aligned}
U_{11} &= 3 \\
U_{12} &= 1 \\
U_{13} &= 1 \\
L_{21}U_{11} &= 1 \\
L_{21}U_{12} + U_{22} &= 3 \\
L_{21}U_{13} + U_{23} &= 1 \\
L_{31}U_{11} &= 1 \\
L_{31}U_{12} + L_{32}U_{22} &= 1 \\
L_{31}U_{13} + L_{32}U_{23} + U_{33} &= 3
\end{aligned}
$$

$$
\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{bmatrix}
$$

Part (c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

$$
\begin{aligned}
U_{11} &= 1 \\
U_{12} &= 1 \\
U_{13} &= 1 \\
L_{21}U_{11} &= 1 \\
L_{21}U_{12} + U_{22} &= 4 \\
L_{21}U_{13} + U_{23} &= 4 \\
L_{31}U_{11} &= 1 \\
L_{31}U_{12} + L_{32}U_{22} &= 4 \\
L_{31}U_{13} + L_{32}U_{23} + U_{33} &= 8
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}
$$

Part (d) Here we are solving $\mathbf{C}\mathbf{C}^{-1} = \mathbf{I}$.

$$
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \therefore [f,\ g,\ h]^\top = [1,\ -1,\ 0]^\top
$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \therefore [x,\ y,\ z]^\top = [4/3,\ -1/3,\ 0]^\top$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \therefore [f,\ g,\ h]^\top = [0,\ 1,\ -1]^\top$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \therefore [x,\ y,\ z]^\top = [-1/3,\ 7/12,\ -1/4]^\top$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \therefore [f,\ g,\ h]^\top = [0,\ 0,\ 1]^\top$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \therefore [x,\ y,\ z]^\top = [0,\ -1/4,\ 1/4]^\top$$

$$\mathbf{C} = \begin{bmatrix} 4/3 & -1/3 & 0 \\ -1/3 & 7/12 & -1/4 \\ 0 & -1/4 & 1/4 \end{bmatrix}$$

Question 3 $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

$$\begin{aligned} U_{11} &= 1 \\ U_{12} &= 1 \\ U_{13} &= 1 \\ L_{21}U_{11} &= 1 \\ L_{21}U_{12} + U_{22} &= 1 \\ L_{21}U_{13} + U_{23} &= 3 \\ L_{31}U_{11} &= 2 \\ L_{31}U_{12} + L_{32}U_{22} &= 5 \\ L_{31}U_{13} + L_{32}U_{23} + U_{33} &= 8 \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

Question 4 A is symmetric.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{U'} = \text{L}^\top$$

Question 5 $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 1 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 1 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \qquad \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - b_2 - b_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - b_2 - b_1 \end{bmatrix}$$

If $b_3 - b_2 - b_1 = 0$ there are an infinite number of solutions, otherwise there are no solutions.

$\mathbf{Ax} = 0.$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So can get solution for any z.