CETI Engineering Maths Notes

Optimisation & Non-linear Equations

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Contents

1 Newton-Raphson: The Square Root in Computing

Calculating the Square Root Find y, the square root of x: $y = \sqrt{x} \Rightarrow f(x) = y^2 - x = 0$

$$\begin{aligned} y_{n+1} &= y_n - \frac{y_n^2 - x}{2y_n} \\ &= y_n \left(1 - \frac{1}{2} + \frac{x}{2y_n^2}\right) \\ &= \frac{y_n}{2}\left(1 + \frac{x}{y_n^2}\right) \end{aligned}$$

Calculating the Inverse Square Root Find y, the inverse square root of x: $y = \frac{1}{\sqrt{x}} \Rightarrow f(x) = \frac{1}{y^2} - x = 0$

$$\begin{aligned} y_{n+1} &= y_n - \frac{1/y_n^2 - x}{-2/y^3} \\ &= y_n \left(1 + \frac{1}{2} - \frac{xy^2}{2}\right) \\ &= \frac{y_n}{2}\left(3 + xy^2\right) \end{aligned}$$

Which method would you use to actually calculate the square root of a number? Calculating the inverse square root using Newton-Raphson and then inverting the answer can often be faster!

Quake's Inverse Square Root The starting value is crucial. Below is the inverse square root algorithm used by Quake. As accuracy is not crucial, only one iteration of Newton-Raphson is required. What makes it fast is the crazy method of approximating an inverse square root by a bit shift and subtraction from a constant!

```
float Q_sqrt( float number )
{
    long i;
    float x2, y;

    y = number;
    i = * ( long * ) &y;
    i = 0x5f3759df - ( i >> 1 );
    y = * ( float * ) &i;
    y = y* ( 1.5F - ( x2 * y * y ) )  // Newton-Raphson

    return y;
}
```

2 Examples Paper 2

Question 1

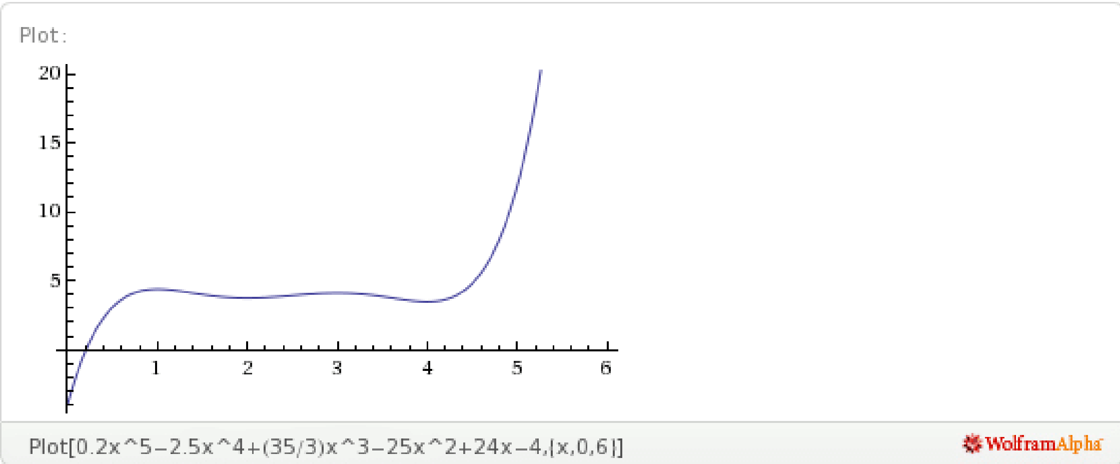
$$f(x) = \frac{1}{5}x^5 - \frac{5}{2}x^4 + \frac{35}{3}x^3 - 25x^2 + 24x - 4 \tag{1}$$
$$f'(x) = (x-1)(x-2)(x-3)(x-4) \tag{2}$$
$$f''(x) = (x-1)(x-2)(2x-7) + (x-3)(x-4)(2x-3) \tag{3}$$

Minima/maxima generally occur at turning points, *i.e.* $f'(x) = 0$. The nature of the points is given by finding the sign of $f''(x)$. if $f''(x) < 0$ then it is a maximum, if $f''(x) > 0$ then it is a minimum.

$$f'(1) = f'(2) = f'(3) = f'(4) = 0$$

x	$f'(x)$	$f''(x)$	$f(x)$
0	24	-50	-4
1	0	-6	$4\frac{11}{30}$
2	0	2	$3\frac{11}{15}$
3	0	-2	$4\frac{1}{10}$
4	0	6	$3\frac{7}{15}$
6	120	154	$75\frac{1}{5}$

So the global maximum and minimum are at the domain boundaries.



Question 2

$$f(x, y) = 3xy - x^3 - y^3 \tag{4}$$

$$f'_x(x, y) = 3y - 3x^2 \qquad f'_y(x, y) = 3x - 3y^2$$
$$f''_x(x, y) = -6x \qquad f''_y(x, y) = -6y$$
$$f''_{yx}(x, y) = 3 \qquad f''_{xy}(x, y) = 3$$

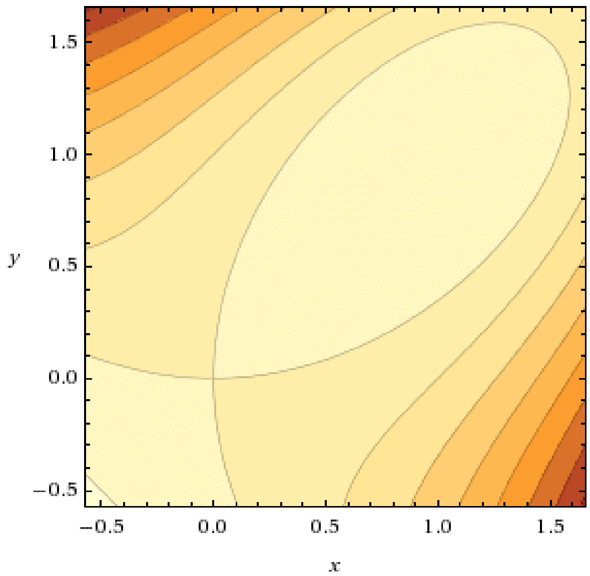
We have just calculated the Hessian Matrix (matrix of second derivatives):

$$H(f) = \begin{bmatrix} f''_x & f''_{xy} \\ f''_{yx} & f''_y \end{bmatrix} = \begin{bmatrix} -6x & 3 \\ 3 & -6y \end{bmatrix}$$

The nature of the points is given by the eigenvalues of the Hessian. For a 2x2 matrix, the determinant is sufficient.
If $f''^2_{xy} - f''_x f''_y > 0$, (x, y) is a saddle point as f''_x and f''_y must have opposing signs.
If $f''^2_{xy} - f''_x f''_y < 0$, (x, y) is a turning point, because f''_x and f''_y must have the same signs.

Solutions ($y = x^2$ and $x = y^2$ give stationary points when the derivatives are zero.):
$x = y = 0$ and $f''^2_{xy} - f''_x f''_y > 0$ (the Hessian is positive-definite) so it is a saddle point.
$x = y = 1$ so $f(1, 1) = 1$ which is a maximum as $f''^2_{xy} - f''_x f''_y < 0$ and $f''_x < 0$.

Contour plot:



ContourPlot[3xy−x^3−y^3]

WolframAlpha

Question 3

$$\min C = 1000P + \frac{4 \times 10^9}{RP} + 2.5 \times 10^5 R$$

$$\frac{\partial C}{\partial P} = 1000 - \frac{4 \times 10^9}{RP^2} = 0$$
$$\frac{\partial C}{\partial R} = 2.5 \times 10^5 - \frac{4 \times 10^9}{R^2 P} = 0$$

$$RP^2 = 4 \times 10^6$$
$$R^2 P = 16000$$

So $P = 1000$ and $R = 4$. Therefore $\min C = 3 \times 10^6$. To confirm this is a minimum, take the second derivatives:

$$H(C) = \begin{bmatrix} C''_P & C''_{PR} \\ C''_{RP} & C''_R \end{bmatrix} = \begin{bmatrix} \frac{8 \times 10^9}{RP^3} & \frac{4 \times 10^9}{R^2 P^2} \\ \frac{4 \times 10^9}{R^2 P^2} & \frac{8 \times 10^9}{R^3 P} \end{bmatrix} = \begin{bmatrix} 2 & 250 \\ 250 & 125000 \end{bmatrix}$$

Negative definite and $C''_P > 0$ so it's a minimum.

Question 4

$$f = (x_1 - 2)^2 + (x_2 - 2)^2$$
$$g = 2x_1 + 3x_2 - 9 = 0$$

g is a constraint on the function f, which we wish to minimise. Construct the Lagrange function Λ:

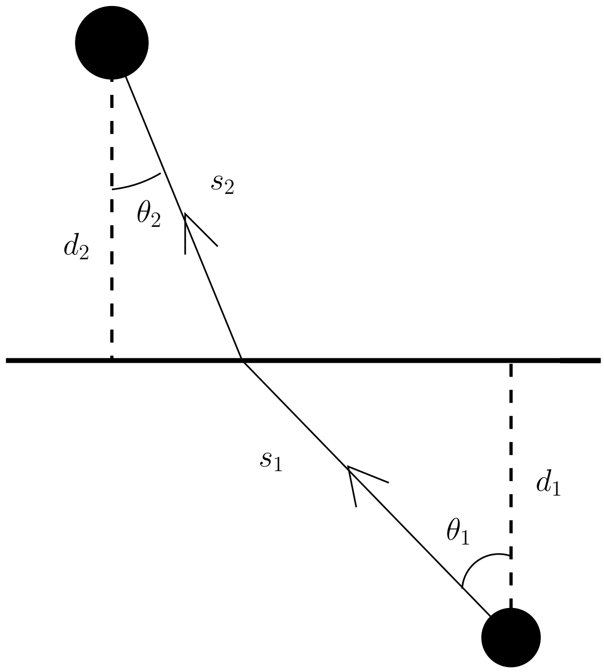
$$\Lambda = (x_1 - 2)^2 + (x_2 - 2)^2 - \lambda(2x_1 + 3x_2 - 9)$$

$$
\begin{aligned}
\frac{\partial \Lambda}{\partial x_1} &= 2x_1 - 4 - 2\lambda = 0 \\
\frac{\partial \Lambda}{\partial x_2} &= 2x_2 - 4 - 3\lambda = 0 \\
\frac{\partial \Lambda}{\partial \lambda} &= 2x_1 + 3x_2 - 9\lambda = 0
\end{aligned}
$$

Solving these equations gives $x_1 = 1.846$, $x_2 = 1.769$ and $\Lambda = -0.154$.
PLOTS

Question 5 The objective function to be minimised is:

$$
\min t = \frac{s_1}{v_1} + \frac{s_2}{v_2} = \frac{d_1}{\cos\theta_1 v_1} + \frac{d_2}{\cos\theta_2 v_2}
$$



The quantity we can change is the point where the runner gets to the water line. This may be expressed as a constraint:

$$
g = x - d_1 \tan\theta_1 - d_2 \tan\theta_2 = 0
$$

The Lagrangian function is:

$$
\Lambda = \frac{d_1}{\cos\theta_1 v_1} + \frac{d_2}{\cos\theta_2 v_2} - \lambda\left(x - d_1\tan\theta_1 - d_2\tan\theta_2\right)
$$

Now differentiate the Lagrangian with respect to θ_1 and θ_2, our two variables:

$$
\begin{aligned}
\frac{\partial \Lambda}{\partial \theta_1} &= \frac{\tan\theta_1 d_1}{\cos\theta_1 v_1} + \frac{\lambda d_1}{\cos^2\theta_1} = \frac{d_1}{\cos^2\theta_1}\left(\frac{\sin\theta_1}{v_1} + \lambda\right) \\
\frac{\partial \Lambda}{\partial \theta_2} &= \frac{\tan\theta_2 d_2}{\cos\theta_2 v_2} + \frac{\lambda d_2}{\cos^2\theta_2} = \frac{d_2}{\cos^2\theta_2}\left(\frac{\sin\theta_2}{v_2} + \lambda\right)
\end{aligned}
$$

So $\dfrac{\partial \Lambda}{\partial \theta_1} = \dfrac{\partial \Lambda}{\partial \theta_2} = 0$ when $\lambda = -\dfrac{\sin\theta_2}{v_2} = -\dfrac{\sin\theta_1}{v_1}$. So we arrive at our result:

$$
\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}
$$

Question 6

$$\begin{aligned} &\max P_w = 3.5A + 5B \\ &2A + B <= 6000 \\ &A + 4B <= 10000 \end{aligned}$$

Therefore $A = B = 2000$ so $P_w = 17000$.
Maybe do this properly with slack variables and the Simplex algorithm:

$$\begin{aligned} P_w - 3.5A - 5B = 0 \\ 2A + B + x = 6000 \\ A + 4B + y = 10000 \end{aligned}$$

$$\left[\begin{array}{cccccc} P & A & B & x & y & RHS \\ \hline 1 & -3.5 & -5 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 6000 \\ 0 & 1 & 4 & 0 & 1 & 10000 \end{array}\right]$$

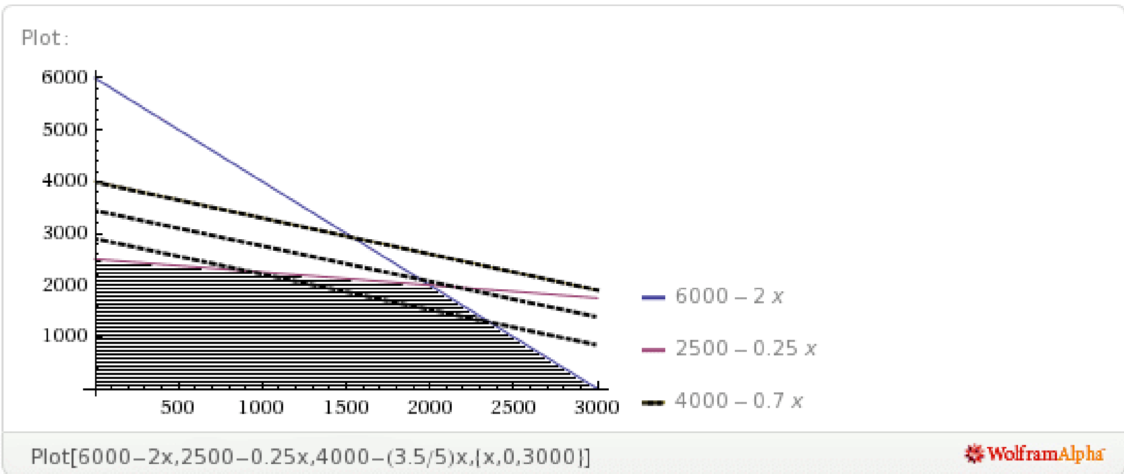
The basic feasible solution is $P = 0, A = 0, B = 0, x = 6000, y = 10000$. Pivot around column A (chosen randomly) and row 2 (because it gives the minimum for A):

$$\left[\begin{array}{cccccc} P & A & B & x & y & RHS \\ \hline 1 & 0 & -3.25 & 7/4 & 0 & 10500 \\ 0 & 1 & 0.5 & 0.5 & 0 & 3000 \\ 0 & 0 & 3.5 & -0.5 & 1 & 7000 \end{array}\right]$$

Now pivot around column B, row 3:

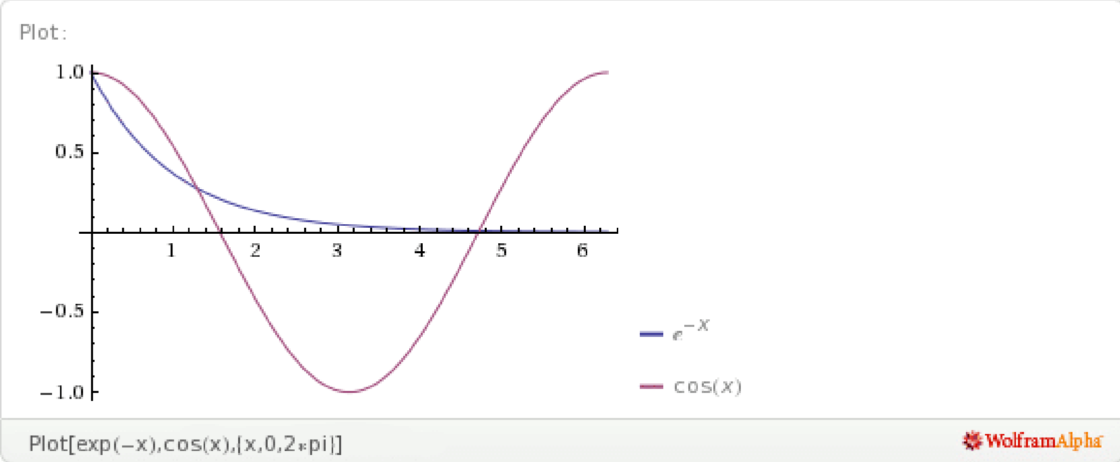
$$\left[\begin{array}{cccccc} P & A & B & x & y & RHS \\ \hline 1 & 0 & 0 & 9/7 & 13/14 & 17000 \\ 0 & 1 & 0 & 4/7 & -1/7 & 2000 \\ 0 & 0 & 1 & -1/7 & 2/7 & 2000 \end{array}\right]$$

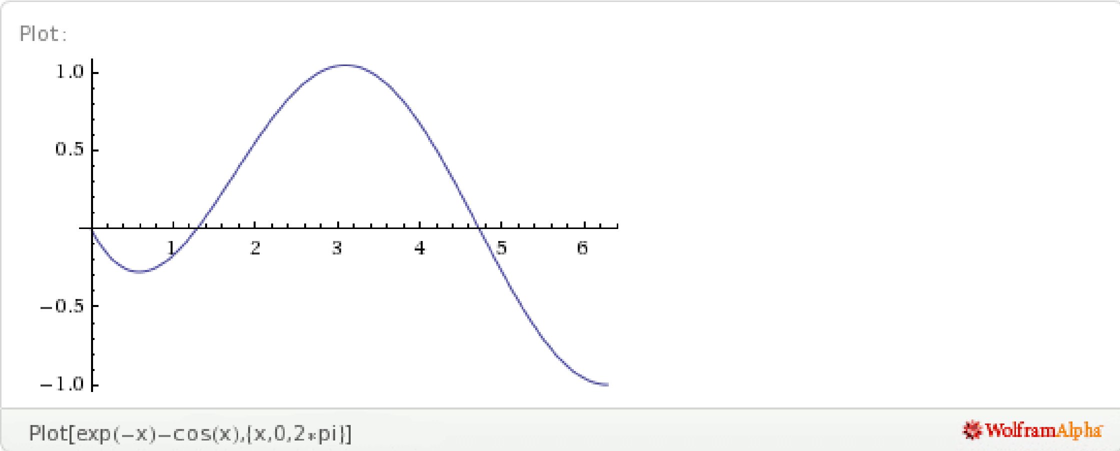
And there we have it, all the values in row 1 are positive, so $P = 17000$ at its maximum.



Question 7

$$g(x) = \exp(-x)\sec x - 1 = 0 \Rightarrow \exp(-x) = \cos x$$





Newton-Raphson: $x_{n+1} = x_n - \dfrac{g(x_n)}{g'(x_n)}$. Need to pick starting value well!
Choose $x_0 = \pi/2$.

$$
\begin{aligned}
x_0 &= \pi/2 \\
x_1 &= \pi/2 - \frac{\exp(-\pi/2) - \cos \pi/2}{\sin \pi/2 - \exp(-\pi/2)} = 1.308 \\
x_2 &= 0.2624 - \frac{\exp(-0.2624) - \cos 0.2624}{\sin 0.2624 - \exp(-0.2624)} = 1.298
\end{aligned}
$$

Using a calculator, after 3 iterations, $x = 1.2927$.

Question 8

$$
\begin{aligned}
f(x, y, z) &= x^2 + y^2 - \exp(-z) = 0 \\
g(x, y, z) &= x + y - z = 0 \\
h(x, y, z) &= x^2 + 2y^2 - \sin\left(\frac{\pi z}{4}\right) = 0
\end{aligned}
$$

Initial point is $(0.4, 0.5, 0.9)$. Find the Jacobian (first derivatives) Matrix for this system:

$$
J = \begin{bmatrix} f'_x & f'_y & f'_z \\ g'_x & g'_y & g'_z \\ h'_x & h'_y & h'_z \end{bmatrix} = \begin{bmatrix} 2x & 2y & \exp(-z) \\ 1 & 1 & -1 \\ 2x & 4y & -\frac{\pi}{4}\cos\left(\frac{\pi z}{4}\right) \end{bmatrix}
$$

In matrix form: $J\mathbf{x} = -\mathbf{f}$ where $\mathbf{x} = (\partial x \ \partial y \ \partial z)^\top$ and $\mathbf{f} = (f \ g \ h)^\top$. Use an iterative solver:

$$\begin{bmatrix} 2x & 2y & \exp(-z) \\ 1 & 1 & -1 \\ 2x & 4y & -\frac{\pi}{4}\cos\left(\frac{\pi z}{4}\right) \end{bmatrix} \begin{bmatrix} \partial x \\ \partial y \\ \partial z \end{bmatrix} = - \begin{bmatrix} f \\ g \\ h \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 1 & 0.40657 \\ 1 & 1 & -1 \\ 0.8 & 2 & -0.59722 \end{bmatrix} \begin{bmatrix} \partial x \\ \partial y \\ \partial z \end{bmatrix} = \begin{bmatrix} -0.00343 \\ 0 \\ -0.01055 \end{bmatrix}$$

Now construct an augmented matrix and perform Gaussian Elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0.8 & 1 & 0.40657 & -0.00343 \\ 0.8 & 2 & -0.59722 & -0.01055 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0.8 & 0.8 & -0.8 & 0 \\ 0 & 0.2 & 1.20657 & -0.00343 \\ 0 & 1.2 & 0.20278 & -0.01055 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0.8 & 0.8 & -0.8 & 0 \\ 0 & 0.2 & 1.20657 & -0.00343 \\ 0 & 0 & -7.03664 & 0.01003 \end{array}\right]$$

Then by back substitution $(\partial x\ \partial y\ \partial z)^\top = (0.00712,\ -0.00855,\ -0.00143)^\top$.
Therefore $x_1 = 0.407, y_1 = 0.491, z_1 = 0.899$.

Question 9

(a)

$$x = \frac{1}{1+x^2}$$

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 0.5 \\ x_2 &= 0.8 \\ x_3 &= 0.610 \\ x_4 &= 0.729 \\ x_5 &= 0.653 \\ x_6 &= 0.701 \end{aligned}$$

(b) For convergence $|g'(x)| < 1$.

$$g'(x) = \frac{-2x}{(x^2+1)^2}$$

Therefore the solution will always convergence:

$$\begin{aligned} \left|\frac{-2x}{(x^2+1)^2}\right| &< 1 \\ 2\,|x| &< \left|(x^2+1)^2\right| \end{aligned}$$

(c)

$$f(x) = x^3 + x - 1 \qquad f'(x) = 2x^2 + 1$$

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{2x_n^2 + 1}$$

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= 0.6667 \\
x_2 &= 0.6863 \\
x_3 &= 0.6814 \\
x_4 &= 0.6826 \\
x_5 &= 0.6823
\end{aligned}$$

(d) A residual is the error in a result, or the change in a solution between successive iterations if the actual answer is not known (often the case!).
The residual for direct substitution is $0.701^3 + 0.701 - 1 = 0.045$.
The residual for Newton-Raphson is $0.6823^3 + 0.6823 - 1 = -6.7 \times 10^{-5}$.

Question 10

$$\begin{aligned}
f(x, y, z) &= xyz - x^2 + y^2 - 1.34 = 0 \\
g(x, y, z) &= xy - z^2 - 0.09 = 0 \\
h(x, y, z) &= \exp(x) - \exp(y) + z - 0.41 = 0
\end{aligned}$$

$x_0 = y_0 = z_0 = 1$. Calculate the Jacobian:

$$J = \begin{bmatrix} f'_x & f'_y & f'_z \\ g'_x & g'_y & g'_z \\ h'_x & h'_y & h'_z \end{bmatrix} = \begin{bmatrix} yz - 2x & xz + 2y & xy \\ y & x & -2z \\ \exp(x) & -\exp(y) & 1 \end{bmatrix}$$

$$\begin{bmatrix} yz - 2x & xz + 2y & xy \\ y & x & -2z \\ \exp(x) & -\exp(y) & 1 \end{bmatrix} \begin{bmatrix} \partial x \\ \partial y \\ \partial z \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$$

For the first iteration:

$$\begin{bmatrix} -1 & 3 & 1 \\ 1 & 1 & -2 \\ 2.718 & -2.718 & 1 \end{bmatrix} \begin{bmatrix} \partial x \\ \partial y \\ \partial z \end{bmatrix} = - \begin{bmatrix} -0.34 \\ -0.09 \\ 0.59 \end{bmatrix}$$

Gaussian Elimination in augmented matrix form:

$$\left[\begin{array}{ccc|c} -1 & 3 & 1 & 0.34 \\ 1 & 1 & -2 & 0.09 \\ 2.718 & -2.718 & 1 & -0.59 \end{array}\right]$$

Pivot around element a_{11}:

$$\left[\begin{array}{ccc|c} -1 & 3 & 1 & 0.34 \\ 0 & 4 & -1 & 0.43 \\ 0 & 5.436 & 3.718 & 0.334 \end{array}\right]$$

Pivot around element a_{22}:

$$\left[\begin{array}{ccc|c} -1 & 3 & 1 & 0.34 \\ 0 & 4 & -1 & 0.43 \\ 0 & 0 & 5.077 & -0.25037 \end{array}\right]$$

Then by back substitution $(\partial x\ \partial y\ \partial z)^\top = (-0.1037,\ 0.0952,\ -0.0493)^\top$.
Therefore $x_1 = 0.896, y_1 = 1.095, z_1 = 0.951$.